7

Dielectrics in the Electrostatic Field

7.1 Introduction

We now know that conductors change the electrostatic field by a mechanism called electrostatic induction, because any conductor has a large number of free charges that move in response to even the slightest electric field.

A wide class of substances known as *dielectrics* or *insulators* do not have free charges inside them. We might expect that, consequently, they can have no effect on the electrostatic field. This is not correct, although the mechanism by which dielectrics affect the electric field is different than in the case of conductors.

Dielectrics or insulators have many applications in electric engineering. Just as there is no electrical device without conductors, there is also no device without insulators. Therefore the analysis of dielectrics in an electrostatic field is as important as that of conductors.

7.2 Polarization of Dielectrics in the Electrostatic Field

Molecules of most substances behave as if electrically neutral when they are not in an electric field. We can imagine a molecule as a positive central point charge Q surrounded by a spherical cloud of negative charges of total charge $-Q$ (Fig. 7.1a). This

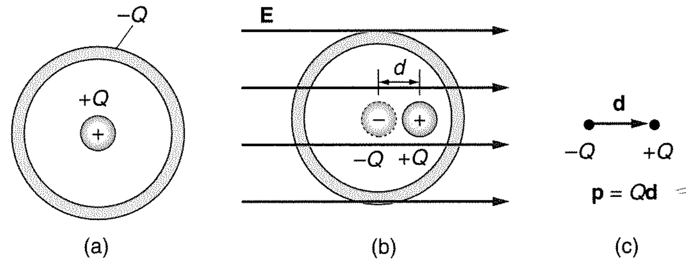

Figure 7.1 (a) Model of a nonpolar molecule, (b) the molecule in an external electric field, and (c) the electric dipole that produces the same field as the molecule in (b)

is an acceptable model, for in reality, at distances larger than a few molecular diameters, the fields of the positive and negative charges cancel out and there is no net electric field. In this rough model of a molecule, some nonelectric forces that keep the molecule spherical and symmetrical must also be present.

Assume now that we move the molecule in Fig. 7.1a into an electric field with electric field strength **E**. The field acts by a force $Q\mathbf{E}$ on the central positive charge, and by the same force, in the opposite direction, on the negatively charged cloud. Due to the forces keeping the molecule together, this will only slightly displace the central positive charge with respect to the center of the negatively charged cloud, as in Fig. 7.1b. The cloud produces the same field at points far away as if the total charge were at its center. Therefore, if we are interested in the electric field produced by the deformed molecule, we can consider it as two point charges, Q and $-Q$, displaced by a small distance d, as in Fig. 7.1c. Two such point charges are known as an *electric dipole*.

In some substances, such as water, the molecules are electric dipoles even with no applied electric field. Such molecules are known as *polar molecules*. Those that are not dipoles in the absence of the field are termed *nonpolar molecules*. In the absence of the electric field, polar molecules are oriented at random and no electric field due to them can be observed. If a polar molecule is brought into an electric field, there are forces on the two dipole charges that tend to align the dipole with the field lines (Fig. 7.2). This alignment is more pronounced for stronger fields.

Thus for dielectrics consisting of any of the two types of molecules, the external electric field makes the substances behave as huge arrays of oriented electric dipoles. We say in such a case that the dielectric is *polarized*. The process of making a dielectric polarized is known as *polarization*.

7.3 The Polarization Vector

According to our model, a polarized dielectric is a vast collection of electric dipoles situated in a vacuum. If we knew the charges Q and $-Q$ of the dipoles and their positions, we could evaluate the electric field strength and the scalar potential at any point. This, however, would be practically impossible due to the extremely large

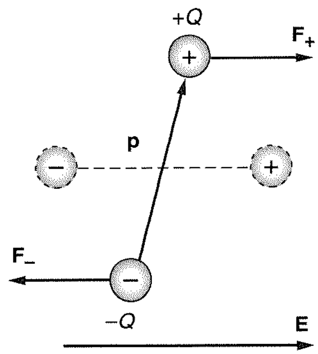

Figure 7.2 Model of a polar molecule in an external electric field

number of dipoles. For this reason we define a kind of average dipole density, a vector quantity known as the *polarization vector*.

We first need to characterize a single dipole by a vector quantity. Let **d** be the position vector of the charge Q of the dipole with respect to the charge $-Q$. We define the *electric dipole moment* of the dipole (Fig. 7.3) as

$$\mathbf{p} = Q\mathbf{d} \qquad (\mathrm{C \cdot m}). \tag{7.1}$$

(Definition of dipole moment)

The unit of **p** is $\mathrm{C \cdot m}$.

Consider now a small volume dv of a polarized dielectric. Let N be the number of dipoles per unit volume inside dv, and **p** be the moment of the dipoles. The polarization vector, **P**, at a point inside dv is defined as

$$\mathbf{P} = \frac{\sum_{dv}\mathbf{p}}{dv} = N\mathbf{p} \qquad (\mathrm{C/m^2}). \tag{7.2}$$

(Definition of the polarization vector)

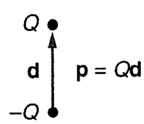

Figure 7.3 The dipole moment of an electric dipole is defined as the product $Q\mathbf{d}$. Note that the vector distance **d** between the two charges is adopted to be directed from the negative to the positive dipole charge.

Because the unit for the dipole moment, $\mathbf{p}$, is C · m, the unit of $\mathbf{P}$ is C/m^2. Note that this is the same unit as that of the surface charge density σ.

From this definition it follows that if we know the polarization vector at a point, we can replace a small volume dv (which contains a large number of dipoles) enclosing that point by a single dipole of moment

$$d\mathbf{p} = \mathbf{P}\, dv \qquad (\mathrm{C \cdot m}). \tag{7.3}$$

(Dipole moment of a small domain dv *with polarization* $\mathbf{P}$*)*

This expression allows us to express the scalar potential and electric field strength of a polarized dielectric as an *integral*.

Equation (7.3) can be used for the evaluation of V and $\mathbf{E}$ of a polarized dielectric, but for that we need to know the expressions for V and $\mathbf{E}$ of a single dipole. Consider the dipole shown in Fig. 7.4. The scalar potential at a point P in the field of a dipole is obtained as the sum of potentials of the two dipole point charges:

$$V_P = \frac{Q}{4\pi\epsilon_0 r_+} + \frac{-Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right). \tag{7.4}$$

Because the distance d between the dipole charges is always much smaller than the distance r of the point P from the dipole, the line segments r, r_+, and r_- are practically parallel. Therefore (Fig. 7.4)

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{r_- - r_+}{r_+ r_-} \simeq \frac{d\cos\theta}{r^2}, \tag{7.5}$$

so that the scalar potential at point P has the form

$$V_P = \frac{Qd\,\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{u}_r}{4\pi\epsilon_0 r^2} \qquad (\mathrm{V}), \tag{7.6}$$

where $\mathbf{u}_r$ is the unit vector directed from the dipole toward point P (see Fig. 7.4). The potential of a point in the field of the dipole does not depend on Q and $\mathbf{d}$ separately,

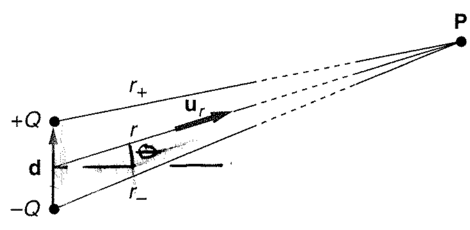

Figure 7.4 A point P in the field of an electric dipole. The distance r between P and the dipole is much larger than the dipole size d

but on their product, **p**, the dipole moment. The electric field strength therefore also depends only on **p**, and not on Q and **d** separately. (It is a simple matter to obtain **E** from the relation $\mathbf{E} = -\nabla V$, which is left as an exercise for the reader.)

The electric scalar potential of a polarized dielectric of volume v is now obtained from Eqs. (7.3) and (7.6) as

$$V = \frac{1}{4\pi\epsilon_0}\int_v \frac{\mathbf{P}\cdot\mathbf{u}_r}{r^2}\,\mathrm{d}v \qquad \text{(V)}. \tag{7.7}$$

(Potential of a polarized dielectric body)

When polarized, a dielectric is a source of an electric field. Consequently, the polarization of a dielectric body depends on the primary field, but also on its own polarization. It can be determined only if we know the dependence of the polarization vector on the *total* electric field strength, **E**. Experiments show that for most substances

$$\mathbf{P} = \chi_e\epsilon_0\,\mathbf{E} \qquad (\mathbf{P} \text{ is in C/m}^2,\ \ \chi_e \text{ is dimensionless}), \tag{7.8}$$

i.e., **P** at every point is proportional to **E** at that point. The constant χ_e is referred to as the *electric susceptibility* of the dielectric. If it is the same at all points, the dielectric is said to be *homogeneous*, and if it varies from point to point, the dielectric is *inhomogeneous*. Dielectrics for which Eq. (7.8) applies are known as *linear dielectrics*, and they are *nonlinear* if such a relation does not hold. For all dielectrics, $\chi_e > 0$. Only for a vacuum, $\chi_e = 0$.

Questions and problems: Q7.1 to Q7.13, P7.1 to P7.3

7.4 Equivalent Charge Distribution of Polarized Dielectrics

A polarized dielectric can always be replaced by an equivalent volume and surface charge distribution in a vacuum. This is a very useful equivalence because we know how to determine the potential and field strength of such a charge distribution. This equivalent charge distribution can be derived from the polarization vector, **P**.

Qualitatively, when a dielectric body is brought into an electric field, as we said earlier, all the molecules become dipoles oriented in the direction of the electric field. Inside a homogeneous dielectric the fields of all the dipoles cancel out on average, because the negative part of one dipole comes close to the positive part of its identical neighbor. However, at the surface of the dielectric there will be ends of dipoles that are uncompensated. This is the extra charge that appears at the surface of a dielectric when brought into an electric field. In the case of homogeneous dielectrics, this is the *only* uncompensated charge due to polarization. Inside an inhomogeneous dielectric, there will be some net volume charge as well, because all the individual dipoles are not identical and their field does not cancel out on average anymore. Both surface and volume polarization charges can now be considered to be in a vacuum, as the rest of the dielectric does not produce any field.

The relationship between the polarization charge inside a closed surface and the polarization vector on the surface can be derived by counting the charge that passes through a surface during the polarization process (the derivation is not given in this text). The resulting expression for the polarization charge in terms of **P** is

$$Q_{\mathrm{p\ in}\ S} = -\oint_S \mathbf{P} \cdot \mathrm{d}\mathbf{S} \qquad \text{(C)}. \tag{7.9}$$

(Polarization (excess) charge in a closed surface enclosing a polarized dielectric)

Example 7.1—Proof that the volume polarization charge density is zero inside a homogeneous polarized dielectric. Consider a polarized *homogeneous* dielectric of electric susceptibility χ_e, *with no volume distribution of free charges*, and a small closed surface ΔS in it. Because we have replaced the dielectric with equivalent charges in a vacuum, Gauss' law applies and the *total* charge, free and polarization, enters on the right-hand side of the formula for Gauss' law. By assumption, there are no free charges in ΔS, and therefore

$$\epsilon_0 \oint_{\Delta S} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = Q_{\mathrm{p\ in}\ \Delta S}. \tag{7.10}$$

According to Eq. (7.9), $Q_{\mathrm{p\ in}\ \Delta S}$ can also be expressed as

$$Q_{\mathrm{p\ in}\ \Delta S} = -\oint_{\Delta S} \mathbf{P} \cdot \mathrm{d}\mathbf{S} = -\chi_e \epsilon_0 \oint_{\Delta S} \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{7.11}$$

Since $\chi_e > 0$, Eqs. (7.10) and (7.11) can both be satisfied only if the flux of **E** through ΔS is zero. The flux of **P** through ΔS is therefore also zero. This means that *inside a homogeneous dielectric there can be no volume distribution of polarization charges, i.e., polarization charges reside only in a thin layer on the dielectric surface.*

Questions and problems: Q7.14 and Q7.15

7.5 Density of Volume and Surface Polarization Charge

Consider now an *inhomogeneous* polarized dielectric. We will show that inside such a dielectric there *is* a volume distribution of polarization charges. To determine the density of these charges, ρ_p, we start from Eq. (7.9). Imagine a small closed surface ΔS enclosing the point at which we wish to determine ρ_p. The left-hand side of Eq. (7.9) can be written as a product of ρ_p and the volume Δv enclosed by ΔS. Consequently,

$$\rho_\mathrm{p} = -\left(\frac{\oint_{\Delta S} \mathbf{P} \cdot \mathrm{d}\mathbf{S}}{\Delta v}\right)_{\Delta v \to 0} \qquad (\mathrm{C/m^3}). \tag{7.12}$$

The expression in parentheses is known as the *divergence* of vector **P**. (For additional explanations of the concept of divergence, read Section A1.4.2 of Appendix 1 before proceeding.) It can always be evaluated from this definition in any coordinate system. In a rectangular coordinate system, the divergence of a vector **P** has the form

$$\text{div}\mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \qquad (\text{C/m}^3), \tag{7.13}$$

(Divergence in a rectangular coordinate system)

where P_x, P_y, and P_z are scalar rectangular components of the vector $\mathbf{P}$. Using the del operator, Eq. (4.20), the expression for the divergence on the right side of this equation can formally be written in a short form,

$$\text{div}\mathbf{P} = \nabla \cdot \mathbf{P}. \tag{7.14}$$

Thus the volume density of polarization charges can be written as

$$\rho_\text{p} = -\text{div}\mathbf{P} = -\nabla \cdot \mathbf{P} \qquad (\text{C/m}^3). \tag{7.15}$$

(Volume density of polarization charges)

Note that Eq. (7.15) is but a shorthand of Eq. (7.12), and that in a rectangular coordinate system, which we will use frequently, $\nabla \cdot \mathbf{P}$ is given by Eq. (7.13).

To determine the density of surface polarization charges, consider Fig. 7.5, showing the interface between two polarized dielectrics, 1 and 2. Apply Eq. (7.9) to the closed surface that looks like a coin, shown in the figure. There is no flux of vector $\mathbf{P}$ through the curved surface because its height approaches zero. Therefore the flux through the closed surface ΔS is given by

$$\oint_{\Delta S} \mathbf{P} \cdot d\mathbf{S} = \mathbf{P}_1 \cdot \Delta\mathbf{S}_1 + \mathbf{P}_2 \cdot \Delta\mathbf{S}_2 \qquad (\text{C}).$$

Let us adopt the reference unit vector, $\mathbf{n}$, normal to the interface, to be directed into dielectric 1 (Fig. 7.5). Then we can write $\Delta\mathbf{S}_1 = \Delta S_1\,\mathbf{n}$ and $\Delta\mathbf{S}_2 = -\Delta S_1\,\mathbf{n}$. The

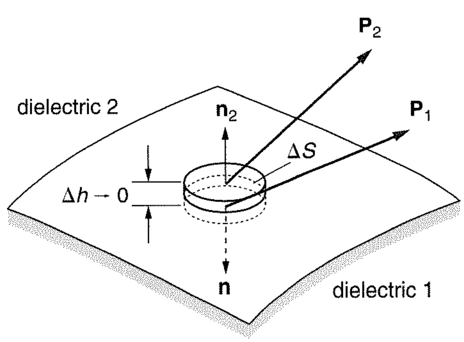

Figure 7.5 Interface between two polarized dielectrics

surface charge density is obtained if we divide the charge enclosed by ΔS by the area ΔS_1 cut out of the interface by ΔS. So we have

$$\sigma_p = \mathbf{n} \cdot (\mathbf{P}_2 - \mathbf{P}_1) \qquad (\mathrm{C/m^2}). \tag{7.16}$$

(Surface density of polarization charges on the interface between two dielectrics)

If we know the polarization vector at all points of a dielectric, from Eq. (7.15) we can find the density of volume polarization charges (if they exist), and from the last equation we can find the density of surface polarization charges (which *always* exist). Because there are no excess charges in the rest of the dielectric, it can be disregarded. The problem of dielectric bodies in electrostatic fields is therefore reduced to that of a *distribution of charges in a vacuum*, a problem we know how to solve. What remains to be done is the determination of the polarization vector at all points. In most instances this is hard to do, but in many important cases it can be done using numerical methods.

Questions and problems: Q7.16 to Q7.19, P7.4 to P7.9

7.6 Generalized Form of Gauss' Law: The Electric Displacement Vector

With the knowledge from the preceding section, Gauss' law can be extended to electrostatic fields with dielectric bodies.

We know that from the electrostatic-field point of view, a polarized dielectric body can be considered as a distribution of volume and surface polarization charges in a vacuum. Gauss' law is valid for a vacuum. Therefore it is straightforward to extend Gauss' law to the case of fields with dielectrics: simply add the polarization charge to the free charge enclosed by S. Consequently, Gauss' law in Eq. (5.4) becomes

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{free in } S} + Q_{\text{polarization in } S}}{\epsilon_0}. \tag{7.17}$$

Usually, this generalized Gauss' law is written in a different form. First, the polarization charge in S is represented as in Eq. (7.9). Note that the surface S is the same for the integral on the left-hand side of Eqs. (7.17) and (7.9). We can, therefore, multiply Eq. (7.17) by ϵ_0, move the integral representing $Q_{\text{polarization in } S}$ to the left-hand side of Eq. (7.17), and use just one integral sign. The result of this manipulation is

$$\oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} = Q_{\text{free in S}} \qquad (\mathrm{C}). \tag{7.18}$$

This is a very interesting result: the flux of the sum of the vectors $\epsilon_0\mathbf{E}$ and $\mathbf{P}$ through any closed surface S is equal to the total *free* charge enclosed by S. The form of Gauss' law (7.18) is more convenient than that of (7.17) because the only charges we can influence directly are free charges.

To simplify Eq. (7.18), we define the *electric displacement vector*, **D**, as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}). \tag{7.19}$$

(Definition of the electric displacement vector)

With this definition, the generalized Gauss' law takes the final form:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{free in S}} \qquad (\mathrm{C}). \tag{7.20}$$

(Generalized Gauss' law)

The expression in Eq. (7.19) is the most general definition of the electric displacement vector. If the dielectric is linear (as most, but not all, dielectrics are), vector **D** can be expressed in terms of the electric field strength, **E**:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}), \tag{7.21}$$

(Electric displacement vector in linear dielectrics)

where

$$\epsilon_r = (1 + \chi_e) \qquad (\text{dimensionless}) \tag{7.22}$$

(Definition of relative permittivity—linear dielectrics only)

is known as the *relative permittivity* of the dielectric, and

$$\epsilon = \epsilon_r \epsilon_0 \qquad (\mathrm{F/m}). \tag{7.23}$$

(Definition of permittivity—linear dielectrics only)

as the *permittivity* of the dielectric.

Because the electric susceptibility, χ_e, is always greater than zero, the relative permittivity, ϵ_r, is always greater than unity. The most frequent values of ϵ_r are between 2 and about 10, but there are dielectrics with much higher relative permittivities. For example, distilled water (which is a dielectric) has relative permittivity of about 80 (this is because its molecules are polar molecules). A table of values of relative permittivities for some common dielectrics is given in Appendix 4.

End class

Skip

Example 7.2—Electric field in a *pn* diode. A *pn* diode, sketched in Fig. 7.6, is a fundamental semiconductor device and is a part of all bipolar transistors. Unlike in a metal, where electrons are the only charge carriers, in a semiconductor diode both negative and positive free charges are responsible for current flow when the diode is biased. The semiconductor material

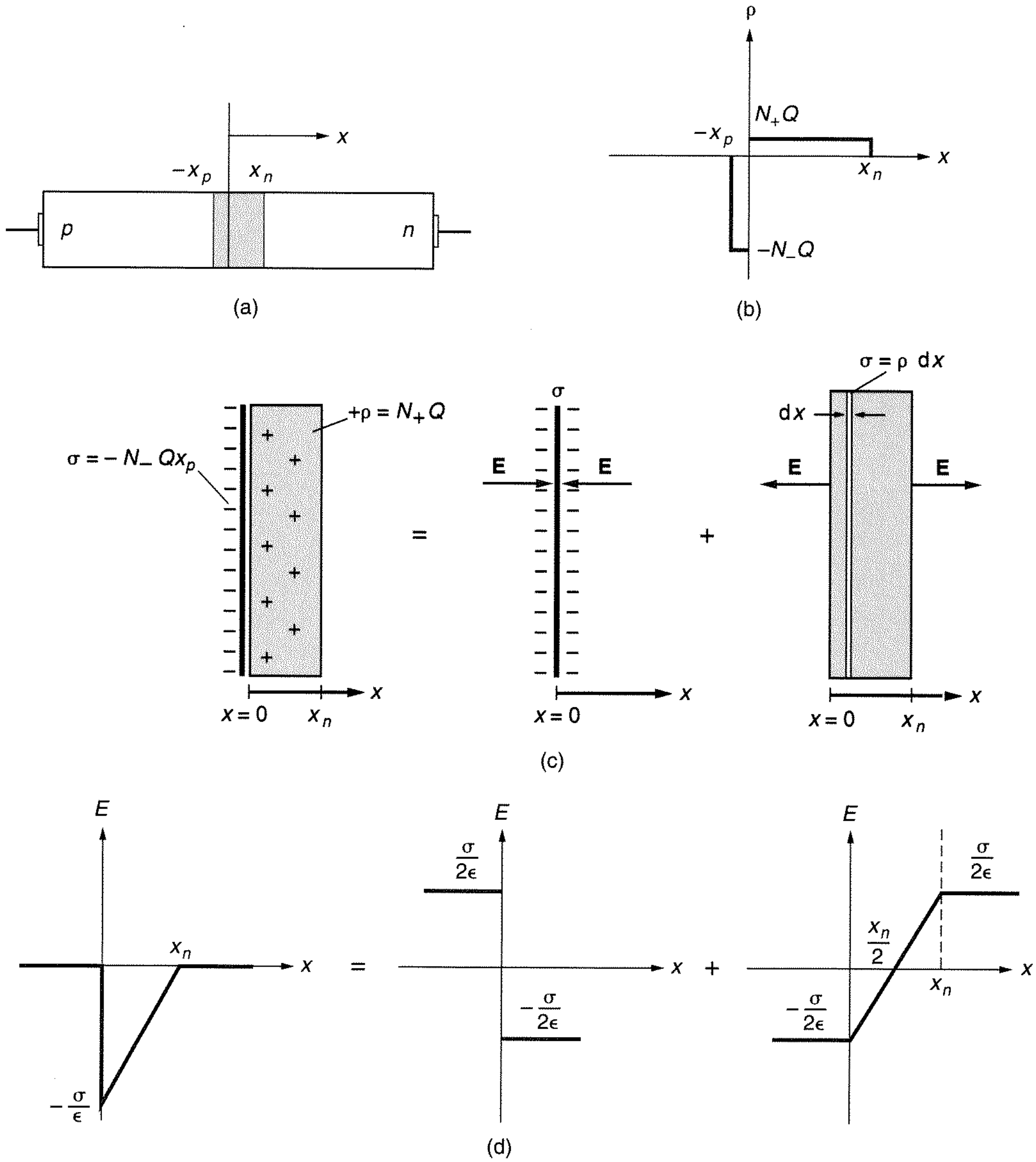

Figure 7.6 (a) Sketch of a *pn* diode and (b) its approximate charge density profile. (c) A diode can be approximated by a sheet of negative surface charge and a bulk of positive volume charge. (d) Superposition of the individual fields of the two charge distributions from (c) gives the final field distribution in the diode.

has a permittivity ϵ (for silicon $\epsilon_r = 11$, and for gallium arsenide $\epsilon_r = 13$), and if it is pure it behaves as a dielectric. When certain impurities called *dopants* are added to the material, it becomes conductive. The p region of the diode is a doped semiconductor material that has p *positive* free charge carriers per unit volume. This part is in physical contact with the n region, which has n *negative* free charge carriers per unit volume.

When the two parts are put together but not biased, the negative charge carriers (electrons) diffuse into the neighboring p region. Positive charge carriers ("holes" with a charge equal to that of an electron) diffuse into the neighboring n region. (The diffusion process is similar to the diffusion of two different gases through a thin membrane, except that the diffused charge carriers remain in the immediate vicinity of the boundary surface.) Because the negative charge carriers move into the region from which positive charge carriers partly left, leaving behind negatively charged atoms, there will be a surplus of negative charge in this thin layer of the p region. Similarly, there will be a surplus of positive charges in the adjoining thin layer of the n region.

These two charged layers produce an electric field (as in a parallel-plate capacitor), resulting in an electric force on free charge carriers that opposes the diffusion process. This electric force eventually (actually, in a very short time) stops the diffusion of free charge carriers. Thin layers on both sides of the boundary surface are thus depleted of their own free charge carriers. These two layers are known as the *depletion region*. Consequently, the depletion region finds itself between the p and n undepleted regions, and contains two layers of equal and opposite charges. Let the number of *positive* charges per unit volume in the n region be N_+, and the number of *negative* charges in the p region be N_-. The volume densities of charge in the two layers of the depletion region are $\rho_+ = N_+Q$ (in the n part), and $\rho_- = -N_-Q$ (in the p part), where Q is the absolute value of the electron charge.

If the diode is not biased (its two terminals are left open), the opposite charges on the two sides of the junction are of equal magnitude. Therefore the thicknesses of the two charged layers, x_p and x_n, are connected by the relation $N_-x_p = N_+x_n$. Usually the diode is made so that the n side of the junction has a much larger concentration of diffused negative free charge carriers than the other, that is, $N_- \gg N_+$. This means that $x_n \gg x_p$. Such a junction is called a one-sided step junction, and its charge concentration profile is sketched in Fig. 7.6b. This tells us that the width of the depletion layer on the p side can be neglected to the first order, i.e., this charged layer can be approximated by a negatively charged sheet of a surface charge density $\sigma = N_-Q/x_p$, Fig. 7.6c. On the n side, the depletion layer is effectively a uniform volume charge density (that is, N_+ is coordinate-independent). We already know from Example 5.3 what the field of the negative surface-charge sheet is, and it is shown in the middle of Fig. 7.6c.

What is the electric field of a volume charge, such as the one on the right in Fig. 7.6c? Outside the charged layer, it is equal to the field of a charged sheet of the same *total* charge:

$$E_{\text{outside}} = \frac{\sigma}{\epsilon} = \frac{\rho x_n}{2\epsilon} = \frac{N_+Qx_n}{2\epsilon}. \tag{7.24}$$

Inside the volume charge, we can apply Gauss' law to a thin slice dx wide, as indicated on the right in Fig. 7.6c, which contains $\rho\,\mathrm{d}x$ surface charge. It is left to the reader to show that integration of the field resulting from all the slices between 0 and x_n gives the following expression for the electric field inside the volume charge density as a function of the x coordinate:

$$E_{\text{inside}} = \frac{\rho}{\epsilon}\left(x - \frac{x_n}{2}\right) = \frac{N_+Qx_n}{\epsilon}\left(x - \frac{x_n}{2}\right). \tag{7.25}$$

DIRECTED IN THE +X DIRECTION IF X>Xn, AND IN THE −X DIRECTION IF X < Xn

This expression is shown graphically on the left in Fig. 7.6d. Using the principle of superposition, we can now add the field of the negative surface charge (in the middle of Fig. 7.6d) to the field of the positive volume charge we found (on the left in Fig. 7.6d) to get the field profile of a *pn* diode, shown on the right in Fig. 7.6d. It is left to the reader as an exercise to sketch the potential distribution inside a diode.

Questions and problems: Q7.20 and Q7.21, P7.10

7.7 Electrostatic Boundary Conditions

In inhomogeneous media consisting of several homogeneous parts there is, obviously, an abrupt change in some quantities describing the field on the two sides of boundaries. For example, if on such a boundary there is a surface polarization charge, it is a source of the electric field component directed in opposite directions on the two sides of the boundary; consequently, the total electric field must have a different direction and magnitude on the two sides of the boundary.

Such abrupt changes of any quantity describing the field must satisfy basic field equations and definitions. Specialized field equations describing this behavior, more precisely connecting the values of any field quantity on two sides of a boundary surface, are known as *boundary conditions*. What are boundary conditions needed for? Note that they represent, in fact, fundamental equations of the electrostatic field specialized to boundary surfaces. Therefore in a medium consisting of several dielectric bodies, the field transition from one body to the adjacent body through a boundary surface *must* be as dictated by the boundary conditions. Otherwise this could not be a real electric field because it would not satisfy the field equations *everywhere*. Note that this is true for all boundary conditions we introduce in later chapters.

Let us apply first the law of conservation of energy of the electrostatic field, Eq. (4.7), to the narrow rectangular contour ΔC in Fig. 7.7. Because the length of the shorter sides approaches zero, the contribution to the line integral of **E** along them is

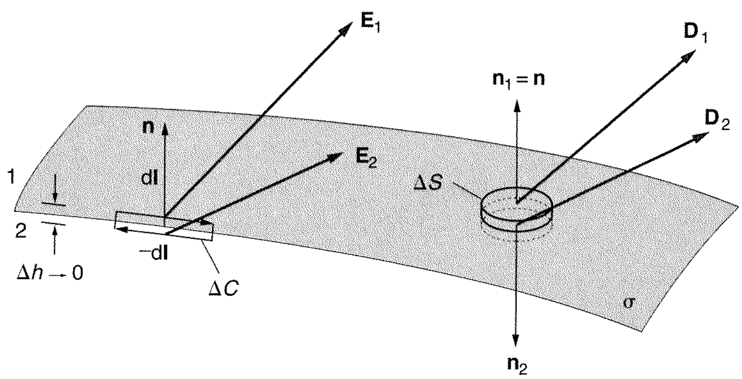

Figure 7.7 Boundary between two media. A narrow rectangular contour is used in the law of conservation of energy and a coinlike closed surface is used in Gauss' law for deriving boundary conditions for vectors **E** and **D**, respectively.

zero. Along the two longer sides, the contribution is $(\mathbf{E}_1 \cdot d\mathbf{l}_1 + \mathbf{E}_2 \cdot d\mathbf{l}_2)$. The scalar products are simply tangential components of the two electric field strength vectors, which we denote by the subscript "t." Because $d\mathbf{l}_2 = -d\mathbf{l}_1$, the boundary condition for the tangential components of vector **E** is

$$E_{1t} = E_{2t} \quad \text{(valid in general).} \tag{7.26}$$

(Boundary condition for tangential components of vector **E***)*

Note that no other assumptions are needed to derive this condition except Eq. (4.7). Consequently, it is valid for all cases of the electrostatic field. We will see that it is valid also for the general case of a time-varying electromagnetic field.

Now let us apply Gauss' law, Eq. (7.20), to the small cylindrical coinlike surface in Fig. 7.7. Let there be a surface charge σ on the boundary inside the surface. There is no flux of vector **D** through the curved surface because its height is vanishingly small. The flux through the two cylinder bases is $D_{1n}\,\Delta S$ (the outward flux) and $-D_{2n}\,\Delta S$ (the inward flux), both with respect to the reference unit vector **n** directed into dielectric 1, where the subscript "n" denotes the normal component. The enclosed charge being $\sigma\,\Delta S$, the generalized Gauss' law yields

$$\mathbf{D}_1 \cdot \mathbf{n} - \mathbf{D}_2 \cdot \mathbf{n} = \sigma, \quad \text{or } D_{1n} - D_{2n} = \sigma \quad \text{(valid in general).} \tag{7.27}$$

(Boundary condition for normal component of vector **D***; unit vector normal,* **n***, directed into medium 1)*

In the special case when there is no surface charge on the boundary, this becomes

$$D_{1n} = D_{2n} \quad \text{(no free surface charges on boundary).} \tag{7.28}$$

Another important case is the boundary between a conductor and a dielectric. Let the dielectric be medium 1, and the conductor be medium 2. We know that there is no field inside a conductor. Therefore $D_{2n} = 0$, and Eq. (7.27) becomes

$$D_n = \sigma \quad \text{(on boundary of dielectric and conductor).} \tag{7.29}$$

Note that this is essentially the same equation as Eq. (6.5). We will see that Eqs. (7.27) to (7.29) are also valid in general, and not only for electrostatic fields.

Questions and problems: Q7.22 to Q7.24, P7.11 to P7.15

7.8 Differential Form of Generalized Gauss' Law

The generalized Gauss' law in Eq. (7.20) can be transformed into a differential equation, known as the differential form of Gauss' law. To obtain this differential equation, let us apply Eq. (7.20) to a small volume Δv enclosed by a surface ΔS, and divide both sides of the equation by Δv. The right side then becomes simply the volume charge density, ρ, inside ΔS. The left side becomes the same as the expression in Eq. (7.12), with **P** substituted by **D**. We know that this expression is the divergence of vector **D**. So we obtain

$$\mathrm{div}\mathbf{D} = \rho. \tag{7.30}$$

(Differential form of generalized Gauss' law)

Since the divergence of **D** is a combination of derivatives of the components of **D**, this is indeed a differential equation in three unknowns, the three scalar components of vector **D**. It is known as a partial differential equation because partial derivatives, with respect to individual coordinates, enter into the equation. We will see that the basic equations of the electromagnetic field, Maxwell's equations, are a set of four partial differential equations. Equation (7.30) is one of these four equations.

7.9 Poisson's and Laplace's Equations: The Laplacian

The potential at a point is related to the volume charge density at that point by a differential equation known as *Poisson's equation*. A special case of Poisson's equation for the case when the volume charge density is zero is called *Laplace's equation*. The derivation of these equations is quite simple.

We know that we can always represent vector **E** as $\mathbf{E} = -\mathrm{grad}\, V = -\nabla V$. For linear media, therefore, $\mathbf{D} = -\epsilon\, \mathrm{grad}\, V = -\epsilon\, \nabla V$, so that from the generalized form of Gauss' law, Eq. (7.13), we obtain

$$\mathrm{div}(\epsilon\, \mathrm{grad} V) = \nabla \cdot (\epsilon \nabla V) = -\rho. \tag{7.31}$$

This is the most general form of Poisson's equation. For the frequent case of a homogeneous dielectric (ϵ the same at all points), Eq. (7.31) becomes

$$\mathrm{div}(\mathrm{grad} V) = \nabla \cdot (\nabla V) = -\frac{\rho}{\epsilon}. \tag{7.32}$$

(Poisson's equation)

Laplace's equation is obtained from Eqs. (7.31) and (7.32) if we set $\rho = 0$:

$$\mathrm{div}(\epsilon\, \mathrm{grad} V) = \nabla \cdot (\epsilon \nabla V) = 0 \tag{7.33}$$

for a general, inhomogeneous dielectric with no free charges, and

$$\mathrm{div}(\mathrm{grad}V) = \nabla \cdot (\nabla V) = 0 \tag{7.34}$$

(Laplace's equation)

for a homogeneous dielectric with no free charges.

The operator div(grad) $= \nabla \cdot \nabla$ is known as *Laplace's operator*, or the *Laplacian*, and is denoted briefly as Δ or ∇^2. It is a simple matter to show that, in a rectangular coordinate system, Laplace's operator has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{7.35}$$

(Laplacian operator in rectangular coordinate system)

As an important example, if the volume charge distribution in a region is a function of a single rectangular coordinate, for example of x, V is then also a function of x only. Poisson's equation becomes

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon}. \tag{7.36}$$

This equation is used often, for example, in the analysis of semiconductor devices including diodes, transistors, and capacitors.

Example 7.3—The *pn* Diode Revisited. In this example, we use Poisson's equation to find the potential distribution in a *pn* diode, using the one-sided step junction approximation from Example 7.2. Poisson's equation for the p side of the junction can be written as

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -\left(-\frac{QN_-}{\epsilon_0\epsilon_r}\right) = \frac{QN_-}{\epsilon_0\epsilon_r}, \tag{7.37}$$

and for the n side as

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -\frac{QN_+}{\epsilon_0\epsilon_r}. \tag{7.38}$$

However, in the one-sided step approximation, the width of the depletion layer on the p side is negligible, so we only need to solve Eq. (7.38). We first integrate once with respect to x from 0 to x. We need one boundary condition to determine the integration constant in this step. We know that there is no electric field outside of the depletion region, so the boundary condition is $\mathrm{d}V/\mathrm{d}x = 0$ at $x = x_n$. Integrating Eq. (7.38) once therefore yields

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{QN_+}{\epsilon_0\epsilon_r}(x - x_n). \tag{7.39}$$

Because we know that $\mathbf{E} = -(\mathrm{d}V/\mathrm{d}x)\mathbf{u}_x$, we can rearrange terms in Eq. (7.39) to obtain the same expression for the electric field as the one shown graphically in Fig. 7.6d. To get the potential, we integrate another time. Let us adopt as the boundary condition that the potential is zero at $x = x_n$ (we know that we can adopt it to be zero at any point). We thus obtain

$$V(x) = -\frac{QN_+x_n^2}{2\epsilon_0\epsilon_r}\left(1 - \frac{x}{x_n}\right)^2. \tag{7.40}$$

As this potential exists inside the diode even when its terminals are not connected to an external voltage source, it is called the *built-in potential*.

When a bias is applied to a diode, it changes the width of the depletion layer. If we connect the diode p region to the positive output of a voltage source and the n side to the negative one, the depletion layer gets narrower, making it easier for free charges to flow through it. This is called *forward bias*. If the diode terminals are connected the other way, the depletion layer becomes thicker and current flow is disabled. This is called *reverse bias*. If an ac voltage is applied to the diode, in one half of the cycle the diode will conduct and in the other half there will be no current. Therefore a diode is a *rectifier*.

Questions and problems: P7.16 to P7.22

7.10 Some Practical Electrical Properties of Dielectrics

Applications of dielectrics in electrical engineering are hardly possible without knowing their electrical properties. We briefly mention here some of these properties.

In addition to relative permittivity, two more properties need particular attention. The first is the *dielectric strength* of a dielectric. This is the largest magnitude of the electric field that can exist in a dielectric without damaging it. If the field magnitude is greater than the dielectric strength of the dielectric, *dielectric breakdown* occurs (the dielectric burns, cracks, ionizes, and becomes conductive, becomes very lossy, etc.).

The typical value of the dielectric strength for air is about $3 \cdot 10^6$ V/m, or 30 kV/cm. For liquid and solid dielectrics the electric field strength ranges from about $15 \cdot 10^6$ V/m to about $40 \cdot 10^6$ V/m. Values of the dielectric strength of some common dielectrics are given in Appendix 4.

Another important property of dielectrics is loss that produces heat. Most dielectrics have a very small number of free charges, so that resistive (Joule's) losses in them due to time-constant fields (except for very large field magnitudes) are usually negligible. In time-varying fields, however, there is a new type of loss, known as the *polarization loss*, that is much larger than Joule's losses. Qualitatively, the time-varying electric field induces time-varying dipoles in the dielectric, which start to vibrate more vigorously due to these oscillations. This vibration is heat, i.e., it represents losses to the field polarizing the dielectric.

Questions and problems: Q7.25 to Q7.27, P7.23

7.11 Chapter Summary

1. If introduced into an electrostatic field, all dielectrics can be visualized as a vast ensemble of small electric dipoles situated in a vacuum. We say that such a dielectric is polarized.
2. The polarization of a dielectric at any point is described by the polarization vector, **P**, representing a vector density of dipole moments at that point. The dipole moment of a dipole of charges Q and $-Q$ separated by a distance **d** (directed from $-Q$ to Q) is defined as $\mathbf{p} = Q\mathbf{d}$.
3. The polarized dielectric can further be considered as an equivalent distribution of volume and surface charges, known as *polarization charges*. These two charge densities are determined in terms of the polarization vector, **P**. The rest of the dielectric has no effect whatsoever on the field and can be removed. The polarization charges must, therefore, be considered to be situated in a vacuum.
4. The vector quantity $\mathbf{D} = (\epsilon_0 \mathbf{E} + \mathbf{P})$ has a simple and useful property: its flux through any closed surface equals the total free charge inside the surface. This equation is known as the *generalized Gauss' law*, and vector **D** as the *electric displacement vector*.
5. The generalized Gauss' law can also be written in the form of a differential equation, $\nabla \cdot \mathbf{D} = \rho$. This is known as the differential form of Gauss' law.
6. There is a simple differential relationship between the potential function at a point and volume charge density at that point, known as the Poisson equation, $\nabla \cdot \epsilon \nabla V = -\rho$. Its special form, when there are no volume charges, is known as Laplace's equation, $\nabla \cdot \epsilon \nabla V = 0$.

QUESTIONS

Q7.1. At a point of a polarized dielectric there are N dipoles per unit volume. Each dipole has a moment **p**. What is the polarization vector at that point?

Q7.2. A body is made of a linear, homogeneous dielectric. Explain what this means.

Q7.3. What is the difference between an inhomogeneous linear dielectric and a homogeneous nonlinear dielectric?

Q7.4. Why is $\chi_e = 0$ for a vacuum?

Q7.5. Are there substances for which $\chi_e < 0$? Explain.

Q7.6. An atom acquires a dipole moment proportional to the electric field strength **E** of the external field, $\mathbf{p} = \alpha \mathbf{E}$ (α is often referred to as the *polarizability*). Determine the electric force on the atom if it is introduced into a *uniform* electric field of intensity **E**.

Q7.7. Answer question Q7.6 for the case in which the atom is introduced into the field of a point charge Q. Determine only the direction of the force, not its magnitude.

Q7.8. A small body—either dielectric or conducting—is introduced into a nonuniform electric field. In which direction (qualitatively) does the force act on the body?

Q7.9. Two point charges are placed near a piece of dielectric. Explain why Coulomb's law cannot be used to determine the *total* force on the two charges.

Q7.10. A small charged body is placed near a large dielectric body. Will there be a force acting between the two bodies? Explain.

Q7.11. A closed surface S situated in a vacuum encloses a total charge Q *and* a polarized dielectric body. Using a sound physical argument, prove that in this case also the flux of the electric field strength vector **E** through S is Q/ϵ_0.

Q7.12. Arbitrary pieces of dielectrics and conductors carrying a total charge Q are introduced through an opening in a hollow, uncharged metal shell. The opening is then closed. Using a physical argument and Gauss' law for a vacuum, prove that the charge appearing on the outer surface of the shell is exactly equal to Q.

Q7.13. A positive point charge is placed in air near the interface of air and a liquid dielectric. Will the interface be deformed? If you think it will be deformed, then will it raise or sink? What if the charge is negative?

Q7.14. Explain in your own words why Eqs. (7.10) and (7.11) imply that the flux of **E** through a closed surface ΔS is zero.

Q7.15. Electric dipoles are arranged along a line (possibly curved) so that the negative charge of one dipole coincides with the positive charge of the next. Describe the electric field of this arrangement of dipoles.

Q7.16. Write Eq. (7.16) for the interface of a dielectric and a vacuum. For case (1) assume the dielectric to be medium 1, and for case (2) medium 2.

Q7.17. Is there a pressure of electrostatic forces acting on a boundary surface between two different dielectrics situated in an electrostatic field? Explain.

Q7.18. Prove that the total polarization charge in any piece of a dielectric material is zero.

Q7.19. A point charge Q is placed inside a spherical metal shell, a distance d from its center. In addition, the shell is filled with an inhomogeneous dielectric. Determine the electric field strength outside the shell.

Q7.20. Does Eq. (7.18) mean exactly the same as Eq. (7.17)? Explain.

Q7.21. Can the relative permittivity of a dielectric be less than one, or negative? Explain.

Q7.22. Can you find an analogy between properly connecting sleeves to a jacket, and using boundary conditions in solving electrostatic field problems? Describe.

Q7.23. Prove that a charged conductor situated in an inhomogeneous but linear dielectric has a potential proportional to its charge. [*Hint:* consider the polarized dielectric as an aggregate of dipoles situated in a vacuum.]

Q7.24. Discuss question Q7.23 for a case in which the dielectric is not linear.

Q7.25. What is the unit of dielectric strength of a dielectric?

Q7.26. Explain how 30 kV/cm is the same as $3 \cdot 10^6$ V/m.

Q7.27. Are polarization losses in a dielectric the same as resistive Joule's losses? Explain.

PROBLEMS

P7.1. Using the relation $\mathbf{E} = -\nabla V$, determine the spherical components E_r, E_θ, and E_ϕ of the electric field strength of the electric dipole in Fig. 7.4.

P7.2. Determine the electric force on a dipole of moment **p** located at a distance r from a point charge Q_0, if the angle between **p** and the direction from the charge is arbitrary.

P7.3. An atom acquires a dipole moment proportional to the electric field strength $\mathbf{E}$ of the external field, $\mathbf{p} = \alpha\mathbf{E}$. Determine the force on the dipole if it is introduced into the field of a point charge Q at a distance r from the charge.

P7.4. A homogeneous dielectric sphere is polarized uniformly over its volume. The polarization vector is $\mathbf{P}$. Determine the distribution of the polarization charges inside and on the surface of the sphere.

P7.5. A thin circular dielectric disk of radius a and thickness d is permanently polarized with a dipole moment per unit volume $\mathbf{P}$, parallel to the axis of the disk that is normal to its plane faces. Determine the electric field strength and the electric scalar potential along the disk axis. Plot your results.

P7.6. Determine the density of volume polarization charges inside a linear but inhomogeneous dielectric of permittivity $\epsilon(x, y, z)$ at a point where the electric field strength is $\mathbf{E}$. There is no volume distribution of free charges inside the dielectric.

P7.7. The permittivity of an infinite dielectric medium is given as the following function of the distance r from the center of symmetry: $\epsilon(r) = \epsilon_0(1 + a/r)$. A small conducting sphere of radius R, carrying a charge Q, is centered at $r = 0$. Determine and plot the electric field strength and the electric scalar potential as functions of r. Determine the volume density of polarization charges.

P7.8. A conducting sphere of radius a carries a charge Q. Exactly one half of the sphere is pressed into a dielectric half-space of permittivity ϵ. Air is above the dielectric. Determine the free and polarization surface charge density on the sphere and in the dielectric.

P7.9. Repeat problem P7.8 for a circular cylinder of radius a with charge Q' per unit length.

P7.10. A small spherical charged body with a charge $Q = -1.9 \cdot 10^{-9}$ C is located at the center of a spherical dielectric body of radius a and relative permittivity $\epsilon_r = 3$. Determine the vectors $\mathbf{E}$, $\mathbf{P}$, and $\mathbf{D}$ at all points, volume and surface density of polarization charges, and the potential at all points. Is it possible to determine the field and potential outside the dielectric body without solving for the field inside the body? Explain.

P7.11. What is $\mathbf{E}$ equal to in a needlelike air cavity inside a homogeneous dielectric of permittivity ϵ if the cavity is parallel to the electric field vector $\mathbf{E}_d$ inside the dielectric (Fig. P7.11)?

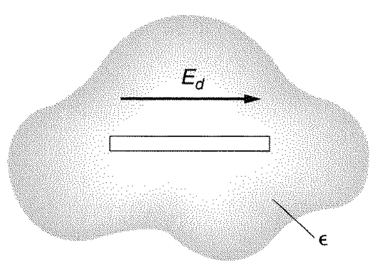

Figure P7.11 A needlelike cavity

P7.12. What is $\mathbf{E}$ equal to in a disklike air cavity with faces normal to the electric field vector $\mathbf{E}_d$ inside a homogeneous dielectric of permittivity ϵ (Fig. P7.12)?

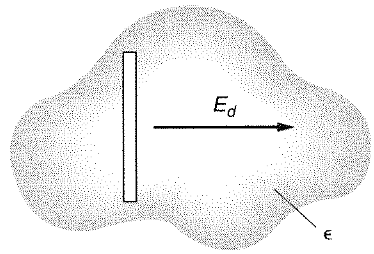

Figure P7.14 A disklike cavity

P7.13. At a point of the boundary surface between dielectrics of permittivities ϵ_1 and ϵ_2, the electric field strength vector in medium 1 makes an angle α_1 with the normal to the boundary, and that in medium 2 an angle α_2. Prove that $\tan\alpha_1/\tan\alpha_2 = \epsilon_1/\epsilon_2$.

P7.14. A dielectric slab of permittivity $\epsilon = 2\epsilon_0$ is situated in a vacuum in an external uniform electric field **E** so that the field lines are perpendicular to the faces of the slab (Fig. P7.14). Sketch the lines of the resulting vectors **E** and **D**.

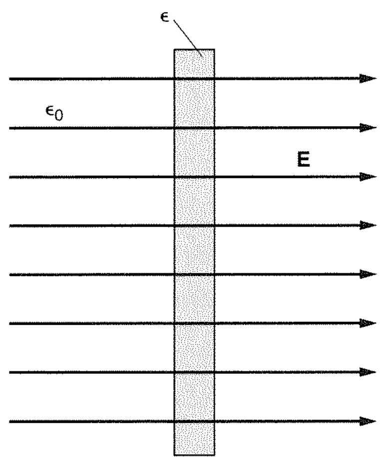

Figure P7.14 Field lines normal to dielectric slab

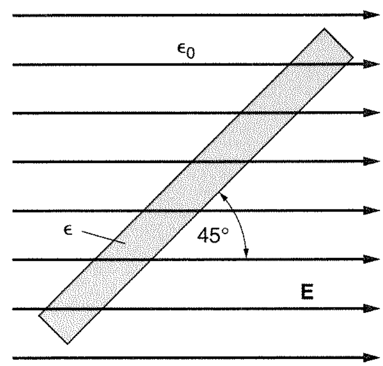

Figure P7.15 Field lines oblique to dielectric slab

P7.15. Repeat problem P7.14 assuming that the dielectric slab is at an angle of 45 degrees with respect to the lines of the external electric field (Fig. P7.15).

P7.16. One of two very large parallel metal plates is at a zero potential, and the other at a potential V. Starting from Laplace's equation, determine the potential, and hence the electric field strength, at all points.

P7.17. Two concentric spherical metal shells, of radii a and b ($b > a$), are at potentials V (the inner shell) and zero. Starting from Laplace's equation in spherical coordinates, determine the potential, and hence the electric field strength, at all points. Plot your results.

P7.18. The charge density at all points between two large parallel flat metal sheets is ρ_0. The sheets are d apart. One of the sheets is at a zero potential, and the other at a potential V. Find the potential at all points between the plates starting from Poisson's equation. Plot your result.

P7.19. Repeat problem P7.18 if the charge density between the plates is $\rho(x) = \rho_0 x/d$, x being a coordinate normal to the plates, with the origin at the zero-potential plate. Plot your result and compare to problem P7.18.

P7.20. Repeat problem P7.19 if the origin is at the plane of symmetry of the system.

P7.21. Two long coaxial cylindrical thin metal tubes of radii a and b ($b > a$) are at potential zero (the outer tube) and V. Starting from Laplace's equation in cylindrical coordinates, determine the potential between the cylinders, and hence the electric field strength.

P7.22. Prove that if V_1 and V_2 are solutions of Laplace's equation, their product is not generally a solution of that equation.

P7.23. The radii of conductors of a coaxial cable with air dielectric are a and b ($b > a$). Determine the maximum value of the potential difference between the conductors for which a complete breakdown of the air dielectric does not occur. The dielectric strength of air is E_0.